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Math 213: Calculus IV

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Quiz # 1, Solutions.

1 - Determine a scalar t such that $\|t(4,3,12)\| = 1$.

$\|t(4,3,12)\| = |t| \cdot \|(4,3,12)\|$ hence any t such that $|t| = 1/\|(4,3,12)\|$ will do. We have $\|(4,3,12)\| = \sqrt{16+9+144} = 13$, so we can choose $t = \frac{1}{13}$ (or $t = \frac{-1}{13}$).

2 - Find the area of the triangle with vertices $(1,-3,7)$, $(2,1,1)$ and $(6,-1,2)$. (Remember, a triangle can be obtained from a parallelogram by cutting it in two.)

The area is $\frac{1}{2}\|\mathbf{F} \times \mathbf{G}\|$, where $\mathbf{F} = (2,1,1) - (1,-3,7) = (1,4,-6)$ and $\mathbf{G} = (6,-1,2) - (1,-3,7) = (5,2,-5)$. We have

$$\mathbf{F} \times \mathbf{G} = (-20+12, -30+5, 2-20) = (-8,-25,-18),$$

and

$$\tfrac{1}{2}\|\mathbf{F} \times \mathbf{G}\| = \tfrac{1}{2}\sqrt{(-8)^2 + (-25)^2 + (-18)^2} = \tfrac{1}{2}\sqrt{1013} \quad (\simeq 15.91).$$

3 - Find a normal (or implicit) equation of the plane containing the three points $(1,-3,7)$, $(2,1,1)$ and $(6,-1,2)$.

Using the normal vector $\mathbf{F} \times \mathbf{G} = (-8,-25,-18)$ computed above, we find the equation $((x,y,z) - (1,-3,7)) \cdot (-8,-25,-18) = 0$ or $(x,y,z) \cdot (-8,-25,-18) = (1,-3,7) \cdot (-8,-25,-18)$, that is,

$$-8x - 25y - 18z = -8 + 75 - 126 = -59.$$

(This can be rewritten as $8x + 25y + 18z = 59$.)

4 - Find a parametric equation of the straight line which contains the point $(1,-3,7)$ and is perpendicular to the plane containing the three points $(1,-3,7)$, $(2,1,1)$ and $(6,-1,2)$. Also find the normal form of this line.

Any point (x,y,z) on this line can be reached by adding to $(1,-3,7)$ a multiple t of the vector $\mathbf{F} \times \mathbf{G} = (-8,-25,-18)$ orthogonal to the plane containing $(1,-3,7)$, $(2,1,1)$ and $(6,-1,2)$. Therefore the parametric equation of the line is

$$(x,y,z) = (1,-3,7) + t(-8,-25,-18).$$

This yields the equations $x = 1 - 8t$, $y = -3 - 25t$ and $z = 7 - 18t$. We find a relation between x and y (that is independent of t) by subtracting 8 times the second equation from 25 times the first:

$$25x - 8y = 25(1-8t) - 8(-3-25t) = 49.$$

Similarly, from the first and third equations we get

$$18x - 8z = 18(1-8t) - 8(7-18t) = -38.$$

Thus the normal equations of the line are $25x - 8y = 49$ and $18x - 8z = -38$.

Answers to additional questions

p.280 # 31. With $\mathbf{F}_i = P_{i+1} - P_i$, $i = 1, \ldots, n-1$ and $\mathbf{F}_n = P_1 - P_n$, we get

$$\mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n = (P_2 - P_1) + (P_3 - P_2) + \cdots + (P_1 - P_n).$$

Each term P_i is added once and subtracted once, so they all cancel out and the sum is $\mathbf{0}$.

p.294 # 34. Assume that $\mathbf{G}$ and $\mathbf{H}$ are not parallel. Then the plane determined by $\mathbf{G}$ and $\mathbf{H}$ is the plane containing $(0,0,0)$ and the endpoints of $\mathbf{G}$ and $\mathbf{H}$. Its parametric equation is $(x, y, z) = s\mathbf{G} + t\mathbf{H}$, and its normal (or implicit) equation is

$$(x, y, z) \cdot (\mathbf{G} \times \mathbf{H}) = 0$$

Now the vector $\mathbf{F} \times (\mathbf{G} \times \mathbf{H})$ is orthogonal to $\mathbf{F}$ and $\mathbf{G} \times \mathbf{H}$. Being orthogonal to $\mathbf{G} \times \mathbf{H}$, it must satisfy

$$(\mathbf{F} \times (\mathbf{G} \times \mathbf{H})) \cdot (\mathbf{G} \times \mathbf{H}) = 0$$

therefore it lies in the plane determined by $\mathbf{G}$ and $\mathbf{H}$.

p.294 # 35. We have $(\mathbf{G} - \mathbf{F}) = (\mathbf{G} - \mathbf{H}) + (\mathbf{H} - \mathbf{F})$ so $(\mathbf{G} - \mathbf{F})$ lies in the plane determined by $(\mathbf{G} - \mathbf{H})$ and $(\mathbf{H} - \mathbf{F})$. Therefore $(\mathbf{G} - \mathbf{F})$ is orthogonal to $(\mathbf{G} - \mathbf{H}) \times (\mathbf{H} - \mathbf{F})$, whence

$$(\mathbf{G} - \mathbf{F}) \cdot ((\mathbf{G} - \mathbf{H}) \times (\mathbf{H} - \mathbf{F})) = 0. \tag{1}$$

According Theorem 7.7, $(\mathbf{G} - \mathbf{H}) \times (\mathbf{H} - \mathbf{F})$ can be expanded using distributivity, as follows:

$$\begin{aligned}(\mathbf{G} - \mathbf{H}) \times (\mathbf{H} - \mathbf{F}) &= (\mathbf{G} \times \mathbf{H}) - (\mathbf{G} \times \mathbf{F}) - (\mathbf{H} \times \mathbf{H}) + (\mathbf{H} \times \mathbf{F}) \\ &= (\mathbf{G} \times \mathbf{H}) - (\mathbf{G} \times \mathbf{F}) + (\mathbf{H} \times \mathbf{F}).\end{aligned}$$

Therefore

$$(\mathbf{G} - \mathbf{F}) \cdot ((\mathbf{G} - \mathbf{H}) \times (\mathbf{H} - \mathbf{F})) = (\mathbf{G} - \mathbf{F}) \cdot ((\mathbf{G} \times \mathbf{H}) - (\mathbf{G} \times \mathbf{F}) + (\mathbf{H} \times \mathbf{F}))$$

and by Theorem 7.4, this expands as

$$\mathbf{G} \cdot (\mathbf{G} \times \mathbf{H}) - \mathbf{G} \cdot (\mathbf{G} \times \mathbf{F}) + \mathbf{G} \cdot (\mathbf{H} \times \mathbf{F}) - \mathbf{F} \cdot (\mathbf{G} \times \mathbf{H}) + \mathbf{F} \cdot (\mathbf{G} \times \mathbf{F}) - \mathbf{F} \cdot (\mathbf{H} \times \mathbf{F}).$$

Now $\mathbf{G} \cdot (\mathbf{G} \times \mathbf{H})$, $\mathbf{G} \cdot (\mathbf{G} \times \mathbf{F})$, $\mathbf{F} \cdot (\mathbf{G} \times \mathbf{F})$ and $\mathbf{F} \cdot (\mathbf{H} \times \mathbf{F})$ are all equal to zero. Thus by 1, we have

$$(\mathbf{G} - \mathbf{F}) \cdot ((\mathbf{G} - \mathbf{H}) \times (\mathbf{H} - \mathbf{F})) = \mathbf{G} \cdot (\mathbf{H} \times \mathbf{F}) - \mathbf{F} \cdot (\mathbf{G} \times \mathbf{H}) = 0.$$

From this follows $\mathbf{G} \cdot (\mathbf{H} \times \mathbf{F}) = \mathbf{F} \cdot (\mathbf{G} \times \mathbf{H})$. The other equalities are proved similarly.