2 pages

Math 213: Calculus IV

May 17/01

Quiz # 2, Solutions.

Let $\mathbf{F}$ be the vector function defined by

$$\mathbf{F}(t) = (\cos(t), 2t, \sin(t)).$$

Determine the velocity, speed and acceleration, the tangential and normal components of the acceleration, the curvature, and the unit tangent, unit normal and binormal vectors of $\mathbf{F}$.

We have $\mathbf{F}'(t) = (-\sin(t), 2, \cos(t))$, $\mathbf{F}''(t) = (-\cos(t), 0, -\sin(t))$, and $\mathbf{F}'(t) \times \mathbf{F}''(t) = (-2\sin(t), -1, 2\cos(t))$. From these we compute the following parameters:

- The velocity $\mathbf{V}(t)$ is just $\mathbf{F}'(t)$, that is, $(-\sin(t), 2, \cos(t))$.
- The speed $v(t)$ is the norm of the velocity, that is,

$$v(t) = \|(-\sin(t), 2, \cos(t))\| = \sqrt{\sin^2(t) + 4 + \cos^2(t)} = \sqrt{5}.$$

- The acceleration $\mathbf{A}(t)$ is just $\mathbf{F}''(t)$, that is, $(-\cos(t), 0, -\sin(t))$.
- The curvature is given by the formula

$$\kappa(t) = \frac{\|\mathbf{F}'(t) \times \mathbf{F}''(t)\|}{\|\mathbf{F}'(t)\|^3} = \frac{\sqrt{4\sin^2(t) + 1 + 4\cos^2(t)}}{\left(\sqrt{5}\right)^3} = \frac{1}{5}.$$

- The tangent component of the acceleration is $a_T(t) = v'(t) = 0$.
- The normal component of the acceleration is $a_N(t) = (v(t))^2 \cdot \kappa(t) = 1$.
- The unit tangent vector is

$$\mathbf{T}(t) = \frac{1}{\|\mathbf{F}'(t)\|}\mathbf{F}'(t) = \frac{1}{\sqrt{5}}(-\sin(t), 2, \cos(t)).$$

- The unit normal vector is

$$\mathbf{N}(t) = \frac{1}{\|\mathbf{T}'(t)\|}\mathbf{T}'(t) = \frac{1}{\|\mathbf{T}'(t)\|}\frac{1}{\sqrt{5}}(-\cos(t), 0, -\sin(t))$$

  with $\|\mathbf{T}'(t)\| = \frac{1}{\sqrt{5}}$, therefore $\mathbf{N}(t) = (-\cos(t), 0, -\sin(t))$.

- The unit binormal vector is

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \frac{1}{\sqrt{5}}(-2\sin(t), -1, 2\cos(t)).$$

Answers to additional questions

p.560 # 15. We have $\mathbf{F}(t) = (\sin(t), \cos(t), 45t)$ so $\mathbf{F}'(t) = (\cos(t), -\sin(t), 45)$. The length function is given by the equation

$$\begin{aligned} s(t) &= \int_0^t \|\mathbf{F}(x)\| dx = \int_0^t \sqrt{\cos^2(x) + \sin^2(x) + 45^2}\, dx \\ &= \int_0^t \sqrt{2026}\, dx = \sqrt{2026}\, x\Big|_0^t = \sqrt{2026}\, t. \end{aligned}$$

Thus we have $t = t(s) = s/\sqrt{2026}$.

Put $\mathbf{G}(s) = \mathbf{F}(t(s)) = \left(\sin\left(s/\sqrt{2026}\right), \cos\left(s/\sqrt{2026}\right), 45\left(s/\sqrt{2026}\right)\right)$. Then $\mathbf{G}'(s) = \left(\cos\left(s/\sqrt{2026}\right), -\sin\left(s/\sqrt{2026}\right), 45/\sqrt{2026}\right)$ and

$$\|\mathbf{G}'(s)\| = \cos^2\left(s/\sqrt{2026}\right) + +\sin^2\left(s/\sqrt{2026}\right) + \left(45/\sqrt{2026}\right)^2 = 1.$$

The length of the derivative of the resulting position vector is indeed 1.

p.560 # 21. The *triple product* $[\mathbf{F}, \mathbf{G}, \mathbf{H}]$ of three vectors is defined on page 293 by the formula

$$[\mathbf{F}, \mathbf{G}, \mathbf{H}] = \mathbf{F} \cdot (\mathbf{G} \times \mathbf{H}).$$

A formula for $\frac{d}{dt}[\mathbf{F}(t), \mathbf{G}(t), \mathbf{H}(t)]$ can be derived from the identities on page 559:

$$\begin{aligned} \frac{d}{dt}[\mathbf{F}(t), \mathbf{G}(t), \mathbf{H}(t)] &= \frac{d}{dt}(\mathbf{F}(t) \cdot (\mathbf{G}(t) \times \mathbf{H}(t))) \\ &= \mathbf{F}'(t) \cdot (\mathbf{G}(t) \times \mathbf{H}(t)) + \mathbf{F}(t) \cdot (\mathbf{G}(t) \times \mathbf{H}(t))' \\ &= \mathbf{F}'(t) \cdot (\mathbf{G}(t) \times \mathbf{H}(t)) + \mathbf{F}(t) \cdot (\mathbf{G}'(t) \times \mathbf{H}(t)) \\ &\quad + \mathbf{F}(t) \cdot (\mathbf{G}(t) \times \mathbf{H}'(t)) \\ &= [\mathbf{F}'(t), \mathbf{G}(t), \mathbf{H}(t)] + [\mathbf{F}(t), \mathbf{G}'(t), \mathbf{H}(t)] \\ &\quad + [\mathbf{F}(t), \mathbf{G}(t), \mathbf{H}'(t)]. \end{aligned}$$

p.570 # 7. $\mathbf{F}(t) = (0, 2\sinh(t), -2\cosh(t))$, $\mathbf{F}'(t) = (0, 2\cosh(t), -2\sinh(t))$, $\mathbf{F}''(t) = (0, 2\sinh(t), -2\cosh(t))$ and $\mathbf{F}'(t) \times \mathbf{F}''(t) = (-4\cosh^2(t) + 4\sinh^2(t), 0, 0) = (-4, 0, 0)$. With this, we get the following parameters: The velocity is just $\mathbf{F}'(t)$, the speed is $v(t) = \|\mathbf{F}'(t)\| = 2\sqrt{\cosh(2t)}$, the acceleration is $\mathbf{F}''(t)$, the curvature is

$$\kappa(t) = \frac{\|\mathbf{F}'(t) \times \mathbf{F}''(t)\|}{\|\mathbf{F}'(t)\|^3} = \frac{1}{2\left(\sqrt{\cosh(2t)}\right)^3},$$

The components of the acceleration are $a_T = v'(t) = 2\sinh(2t)/\sqrt{\cosh(2t)}$ and $a_N = v^2(t)\kappa(t) = 2/\sqrt{\cosh(2t)}$, the unit tangent vector is $\mathbf{T} = \frac{1}{\|\mathbf{F}'(t)\|}\mathbf{F}'(t)$, the unit binormal vector is $\mathbf{B} = \frac{1}{\|\mathbf{F}'(t) \times \mathbf{F}''(t)\|}\mathbf{F}'(t) \times \mathbf{F}''(t) = (-1, 0, 0)$ and the unit normal vector is $\mathbf{N} = \mathbf{B} \times \mathbf{T} = \frac{1}{2\sqrt{\cosh(2t)}}(0, -2\sinh(t), -2\cosh(t))$.