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Math 213: Calculus IV

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Quiz # 3, Solutions

1 - Let $\mathbf{F}$ be the vector field defined by $\mathbf{F}(x, y, z) = (1, 2x, 3x^2)$. Verify that the curve defined by $\mathbf{C}(t) = (t+2, t^2, t^3)$ is a streamline of $\mathbf{F}$ through the point $(4, 4, 8)$.
(Recall that $\mathbf{C}$ is a streamline of $\mathbf{F}$ if and only if $\mathbf{F}(x, y, z)$ is tangent to the curve $\mathbf{C}$ at each point (x, y, z) on $\mathbf{C}$.)

We have $\mathbf{C}'(t) = (1, 2t, 3t^2)$; so $\mathbf{C}$ is a streamline of $\mathbf{F}$ if $\mathbf{F}(\mathbf{C}(t))$ is a multiple of $(1, 2t, 3t^2)$ for all values of t. However,

$$\mathbf{F}(\mathbf{C}(t) = \mathbf{F}(t+2, t^2, t^3) = (1, 2(t+2), 3(t+2)^2),$$

and $t = 0$, this gives $\mathbf{C}'(0) = (1, 0, 0)$ and $\mathbf{F}(\mathbf{C}(0) = (1, 4, 12)$. Since $(1, 4, 12)$ is not a multiple of $(1, 0, 0)$, we conclude that $\mathbf{C}$ is **not** a streamline of $\mathbf{F}$.

The streamlines of $\mathbf{F}$ are found through the equations

$$\frac{dx}{1} = \frac{dy}{2x}, \frac{dx}{1} = \frac{dz}{3x^2}.$$

The first of these can be written $2x\,dx = dy$, and integrating both sides gives $x^2 = y + c$, where c is an arbitrary constant. Similarly, the second equation is $3x^2 dx = dz$ which yields $x^3 = z + d$ for some constant d. Now for $(x, y, z) = (4, 4, 8)$, the first equation gives $4^2 = 4 + c$ whence $c = 12$, and the second equation gives $4^3 = 8 + d$ whence $d = 56$. The streamline of F through $(4, 4, 8)$ satisfies the equations $y = x^2 - 12$ and $z = x^3 - 56$. Putting $t = x$, we get the parametrised form

$$\mathbf{C}(t) = (t, t^2 - 12, t^3 - 56).$$

(Indeed, we then have $\mathbf{C}'(t) = (1, 2t, 3t^2) = \mathbf{F}(\mathbf{C}(t))$.)

2 - Let $f(x, y, z) = xyz$. Compute the gradient of f and use it to compute the directional derivative of f at the point $(1, 0, 2)$ in the direction of the unit vector $D = (\frac{4}{13}, \frac{12}{13}, \frac{3}{13})$.

Does there exist a unit vector D such that the directional derivative of f at $(1, 0, 2)$ in the direction of D is 3?

We have $\nabla f(x, y, z) = (yz, xz, xy)$, thus $\nabla f(1, 0, 2) = (0, 2, 0)$. The directional derivative of f in the direction of $(\frac{4}{13}, \frac{12}{13}, \frac{3}{13})$ is

$$(0, 2, 0) \cdot (\tfrac{4}{13}, \tfrac{12}{13}, \tfrac{3}{13}) = \tfrac{24}{13}.$$

Furthermore, the maximum possible value of the directional derivative of f in the direction of a unit vector is

$$\|\nabla f(1, 0, 2)\| = \|(0, 2, 0)\| = 2.$$

Therefore there does not exist a unit vector D such that the directional derivative of f at $(1, 0, 2)$ in the direction of D is 3.

3 - Compute the gradient of the function $g(x, y, z) = xz - y^2$ and use it to find the equation of the tangent plane to the surface $xz = y^2$ at the point $(x, y, z) = (3, 6, 12)$.

We have $\nabla f(x, y, z) = (z, -2y, x)$, and $\nabla f(3, 6, 12) = (12, -12, 3)$. The tangent plane to the surface $xz = y^2$ at the point $(3, 6, 12)$ is given in implicit form by the equation

$$((x, y, z) - (3, 6, 12)) \cdot (12, -12, 3) = 0;$$

Which reduces to

$$4x - 4y + z = 0.$$

Answers to additional questions

p.588 # 8. $\phi(x, y, z) = -\frac{1}{||\mathbf{R}||}$ with $\mathbf{R}(x, y, z) = (x, y, z)$.

Here the fancy way to compute the gradient uses two identities:

$$\begin{aligned} \nabla (fg) &= f\nabla g + g\nabla f. && (1) \\ \nabla (\mathbf{R} \cdot \mathbf{R}) &= \nabla(x^2 + y^2 + z^2) = 2\mathbf{R}. && (2) \end{aligned}$$

In our particular example, we have $\phi(x, y, z)||\mathbf{R}|| = -1$. Squaring both sides yields $\phi^2(x, y, z)\,(\mathbf{R} \cdot \mathbf{R}) = 1$, and we can compute the gradient implicitly:

$$\begin{aligned} \nabla \left[\phi^2(x, y, z)\,(\mathbf{R} \cdot \mathbf{R})\right] &= (0, 0, 0) \\ \phi^2(x, y, z)\nabla\left[\mathbf{R} \cdot \mathbf{R}\right] + (\mathbf{R} \cdot \mathbf{R})\,\nabla\left[\phi^2(x, y, z)\right] &= (0, 0, 0) \\ \phi^2(x, y, z)\left[2\mathbf{R}\right] + (\mathbf{R} \cdot \mathbf{R})\left[2\phi(x, y, z)\nabla\phi(x, y, z)\right] &= (0, 0, 0) \end{aligned}$$

Solving for $\nabla\phi(x, y, z)$, with $\phi(x, y, z) = -\frac{1}{||\mathbf{R}||}$ and $\mathbf{R} \cdot \mathbf{R} = ||\mathbf{R}||^2$, we get

$$\nabla\phi(x, y, z) = \frac{-2\phi^2(x, y, z)}{2||\mathbf{R}||^2\phi(x, y, z)}\mathbf{R} = \frac{1}{||\mathbf{R}||^3}\mathbf{R}.$$

In particular $\nabla\phi(-2, 1, 1) = \frac{1}{6\sqrt{6}}(-2, 1, 1)$.

p.589 # 21. The two surfaces are given by the equations $f(x, y, z) = 0$ and $g(x, y, z) = 0$, where $f(x, y, z) = 3x^2 + 2y^2 - z$ and $g(x, y, z) = -2x + 7y^2 - z$. We have

$$\nabla f(x, y, z) = (6x, 4y, -1), \nabla f(1, 1, 5) = (6, 4, -1)$$

and

$$\nabla g(x, y, z) = (-2, 14y, -1), \nabla g(1, 1, 5) = (-2, 14, -1).$$

The angle θ between the two surfaces at the point $(1, 1, 5)$ is equal to the angle between the two gradient vectors which can be computed as follows

$$\cos(\theta) = \frac{(6, 4, -1) \cdot (-2, 14, -1)}{||(6, 4, -1)||\;||(-2, 14, -1)||} = \frac{45}{\sqrt{53}\,\sqrt{201}},$$

$$\theta = \arccos\left(\frac{45}{\sqrt{53}\,\sqrt{201}}\right) \simeq 1.12 \simeq 0.36\pi \text{ radians (or 64.15 degrees)}.$$