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Math 213: Calculus IV

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Quiz # 4, Solutions.

1 - For $\mathbf{F}(x,y,z) = (y^2z, z^2x, x^2y)$, evaluate the curl $\nabla \times \mathbf{F}$, and then verify that the divergence of $\nabla \times \mathbf{F}$ is 0.

The curl $\nabla \times \mathbf{F}$ is given by

$$\begin{aligned} \nabla \times \mathbf{F}(x,y,z) &= \left(\frac{\delta}{\delta y}x^2y - \frac{\delta}{\delta z}z^2x, \frac{\delta}{\delta z}y^2z - \frac{\delta}{\delta x}x^2y, \frac{\delta}{\delta x}z^2x - \frac{\delta}{\delta y}y^2z\right) \\ &= \left(x^2 - 2xz, y^2 - 2xy, z^2 - 2yz\right), \end{aligned}$$

and the divergence of $\nabla \times \mathbf{F}$ is

$$\begin{aligned} \nabla \cdot (\nabla \times \mathbf{F}(x,y,z)) &= \frac{\delta}{\delta x}\left(x^2 - 2xz\right) + \frac{\delta}{\delta y}\left(y^2 - 2xy\right) + \frac{\delta}{\delta z}\left(z^2 - 2yz\right) \\ &= 2x - 2z + 2y - 2x + 2z - 2y = 0. \end{aligned}$$

2 - Find the center of mass of a string lying along the semicircle

$$\mathbf{C}(t) = (\cos(t), \sin(t)), 0 \le t \le \pi,$$

with uniform unit density $f(x,y) = 1$.

The mass m of the string is $\int_{\mathbf{C}} 1\ ds$. With the given parametrisation, we have $\mathbf{C}'(t) = (-\sin(t), \cos(t))$ and $\|\mathbf{C}'(t)\| = \sqrt{\sin^2(t) + \cos^2(t)} = 1$. Therefore, $ds = \|\mathbf{C}'(t)\|dt = dt$ and

$$m = \int_0^\pi 1dt = t|_0^\pi = \pi.$$

The center of mass $(\overline{x}, \overline{y})$ of the string is given by the equations

$$\begin{aligned} \overline{x} &= \frac{1}{m}\int_{\mathbf{C}} x \cdot 1\ ds = \frac{1}{\pi}\int_0^\pi \cos(t)dt = \frac{1}{\pi}\sin(t)\Big|_0^\pi = 0, \\ \overline{y} &= \frac{1}{m}\int_{\mathbf{C}} y \cdot 1\ ds = \frac{1}{\pi}\int_0^\pi \sin(t)dt = \frac{-1}{\pi}\cos(t)\Big|_0^\pi = \frac{2}{\pi}. \end{aligned}$$

Therefore the center of mass of the string is the point $(0, \frac{2}{\pi})$.

3 - Evaluate $\int_0^3 \left(\int_{-\pi/2}^{\pi/2} \cos(y)dy\right) dx$ and then explain how it is related to the work done by the force $\mathbf{F}(x,y) = (x\sin(x), x\cos(y))$ on a particle moving counterclockwise along the rectangle with vertices $(0, -\pi/2)$, $(3, -\pi/2)$, $(3, \pi/2)$ and $(0, \pi/2)$.

$$\int_0^3 \left(\int_{-\pi/2}^{\pi/2} \cos(y)dy\right) dx = \int_0^3 \left(\sin(y)|_{-\pi/2}^{\pi/2}\right) dx = \int_0^3 2dx = 6. \qquad (1)$$

Let $\mathbf{C}(t)$ be a parametrization of the rectangle in the positive direction. Then the work of the force $\mathbf{F}$ on the particle is $\oint_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{C}$ and by Green's theorem we have

$$\oint_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{C} = \int\int_D \frac{\delta}{\delta x} g - \frac{\delta}{\delta y} f \; dy \; dx$$

where D is the interior of $\mathbf{C}$, described by the inequalities $0 \leq x \leq 3$, $-\pi/2 \leq y \leq \pi/2$. Here, $g(x,y) = x\cos(y)$, $f(x,y) = x\sin(x)$ whence $\frac{\delta}{\delta x} g - \frac{\delta}{\delta y} f = \cos(y)$ and

$$\int\int_D \frac{\delta}{\delta x} g - \frac{\delta}{\delta y} f \; dy \; dx = \int_0^3 \int_{-\pi/2}^{\pi/2} \cos(y) \; dy \; dx.$$

The integral (1) computes the work of $\mathbf{F}$ on the particle along its path.

Answers to additional questions

p.606 # 28. The lengths of the sides of a rectangle with vertices $(1,4,3)$, $(1,1,3)$, $(6,1,3)$ and $(6,4,3)$ are in order $3, 5, 3$ and 5 units. If the densities of the corresponding sides are respectively $3, 3, 5$ and 5 grams per unit, then the total mass of the wire is $3 \cdot 3 + 5 \cdot 3 + 3 \cdot 5 + 5 \cdot 5 = 64$ grams. Moreover, each side being of uniform density, their center of mass must lie at their midpoint, namely the points $\left(1, \frac{5}{2}, 3\right)$, $\left(\frac{7}{2}, 1, 3\right)$, $\left(6, \frac{5}{2}, 3\right)$, $\left(\frac{7}{2}, 4, 3\right)$. The center of mass of the whole wire is then obtained as the weighted average

$$\frac{1}{64}\left(9\left(1, \frac{5}{2}, 3\right) + 15\left(\frac{7}{2}, 1, 3\right) + 15\left(6, \frac{5}{2}, 3\right) + 25\left(\frac{7}{2}, 4, 3\right)\right) = \left(\frac{239}{64}, \frac{175}{64}, 3\right)$$

p.610 # 15. Let $\mathbf{C}$ be a positively oriented simple closed curve with interior D. Then the area of D is $\int\int_D 1 \; dxdy$ and by Green's theorem, this can be rewritten as

$$\int\int_D 1 \; dxdy = \oint_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{C} = \oint_{\mathbf{C}} f(x,y)dx + g(x,y)dy$$

for any $\mathbf{F}(x,y) = (f(x,y), g(x,y))$ such that $\frac{\delta}{\delta x} g - \frac{\delta}{\delta y} f = 1$. In particular:

(a) With $\mathbf{F}(x,y) = (-y, 0)$, we see that the area of D is $\oint_{\mathbf{C}} -y dx$.

(b) With $\mathbf{F}(x,y) = (0, x)$, we see that the area of D is $\oint_{\mathbf{C}} x dy$.

Of course, this implies that the average of these two integrals is also the area of D.

p.610 # 17. Let $u(x,y)$ be a continuous scalar field with continuous first and second partial derivatives on a simple closed path $\mathbf{C}$ and throughout the interior D of $\mathbf{C}$. Define $\mathbf{F}(x,y) = \left(-\frac{\delta}{\delta y} u(x,y), \frac{\delta}{\delta x} u(x,y)\right)$. Then by Green's theorem we have

$$\begin{aligned}
\oint_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{C} &= \oint_{\mathbf{C}} -\frac{\delta}{\delta y} u(x,y)dx + \frac{\delta}{\delta x} u(x,y)dy \\
&= \int\int_D \left(\frac{\delta}{\delta x}\frac{\delta}{\delta x} u(x,y) - \frac{\delta}{\delta y}(-1)\frac{\delta}{\delta y} u(x,y)\right) dxdy \\
&= \int\int_D \left(\frac{\delta^2}{\delta x^2} u(x,y) + \frac{\delta^2}{\delta y^2} u(x,y)\right) dxdy.
\end{aligned}$$