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Math 213: Calculus IV

June 7/01

Quiz # 5, June 7 2001.

Name:

Student number:

1 - Find the mass and center of mass of the hemisphere $x^2+y^2+z^2 = 1, z \geq 0$.

Here the density is assumed to be uniform $f(x, y, z) = 1$ everywhere, and the surface Σ of the sphere is given by the equation $z = S(x, y) = \sqrt{1-x^2-y^2}$, where $x^2 + y^2 \leq 1$. We have

$$\frac{\delta S}{\delta x} = \frac{-x}{\sqrt{1-x^2-y^2}}, \ \frac{\delta S}{\delta y} = \frac{-y}{\sqrt{1-x^2-y^2}}.$$

Therefore

$$\begin{aligned} m &= \int\int_{\Sigma} 1\, d\sigma = \int\int_{x^2+y^2\leq 1} \sqrt{\left(\frac{\delta S}{\delta x}\right)^2 + \left(\frac{\delta S}{\delta y}\right)^2 + 1}\, dxdy \\ &= \int\int_{x^2+y^2\leq 1} \sqrt{\frac{x^2}{1-x^2-y^2} + \frac{y^2}{1-x^2-y^2} + 1}\, dxdy \\ &= \int\int_{x^2+y^2\leq 1} \frac{1}{\sqrt{1-x^2-y^2}}\, dxdy \end{aligned}$$

With the polar substitutions $x = r\cos(\theta)$ $y = r\sin(\theta)$, $dxdy = rdrd\theta$, this becomes

$$m = \int_0^{2\pi}\int_0^1 \frac{1}{\sqrt{1-r^2}} rdrd\theta = \int_0^{2\pi} -\sqrt{1-r^2}\Big|_0^1 d\theta = \int_0^{2\pi} 1\ d\theta = 2\pi.$$

The center of mass $(\overline{x}, \overline{y}, \overline{z})$ satisfies the equations

$$\overline{x} = \tfrac{1}{m}\int\int_{x^2+y^2\leq 1} \frac{x}{\sqrt{1-x^2-y^2}}\, dxdy, \ \overline{y} = \tfrac{1}{m}\int\int_{x^2+y^2\leq 1} \frac{y}{\sqrt{1-x^2-y^2}}\, dxdy,$$

$$\overline{z} = \tfrac{1}{m}\int\int_{x^2+y^2\leq 1} \frac{z}{\sqrt{1-x^2-y^2}}\, dxdy$$

Obviously, we must have $\overline{x} = \overline{y} = 0$ since the hemisphere is symmetric about the z axis. This is also verified easily through integration in polar coordinates, by interchanging the order of integration:

$$\begin{aligned} \overline{x} &= \tfrac{1}{m}\int_0^{2\pi}\int_0^1 \frac{r\cos(\theta)}{\sqrt{1-r^2}}\, rdrd\theta = \tfrac{1}{m}\int_0^1\int_0^{2\pi} \left(\frac{r^2}{\sqrt{1-r^2}}\right)\cos(\theta)d\theta dr \\ &= \tfrac{1}{m}\int_0^1 \left(\frac{r^2}{\sqrt{1-r^2}}\right)\sin(\theta)\Big|_0^{2\pi} dr = \tfrac{1}{m}\int_0^1 0dr = 0. \end{aligned}$$

and similarly for $\overline{y}$. With $z = \sqrt{1-x^2-y^2}$ we get

$$\overline{z} = \tfrac{1}{m}\int\int_{x^2+y^2\leq 1} \frac{z}{\sqrt{1-x^2-y^2}}\, dxdy = \tfrac{1}{m}\int\int_{x^2+y^2\leq 1} 1\ dxdy = \tfrac{1}{2\pi}\pi = \tfrac{1}{2}.$$

Thus the center of mass of the hemisphere is the point $(0, 0, \frac{1}{2})$.

2 - Find a constant n for which the vector field

$$\mathbf{F}(x, y) = (y^n + 2xy^3, 2xy + 3x^2y^2 + 4y^3)$$

is conservative. For this constant n, find a potential function ϕ such that $\nabla\phi = \mathbf{F}$ and use it to find the work done by $\mathbf{F}$ on a particle moving on any path from $(1, -1)$ to $(2, 1)$.

We have

$$\frac{\delta}{\delta y}\left(y^n + 2xy^3\right) = ny^{n-1} + 6xy^2, \ \frac{\delta}{\delta x}\left(2xy + 3x^2y^2 + 4y^3\right) = 2y + 6xy^2.$$

These two are equal only if $ny^{n-1} = 2y$, that is, $n = 2$. Thus we put

$$\mathbf{F}(x, y) = (y^2 + 2xy^3, 2xy + 3x^2y^2 + 4y^3) = \nabla\phi(x, y) = \left(\frac{\delta\phi}{\delta x}, \frac{\delta\phi}{\delta y}\right).$$

We then have

$$\phi(x, y) = \int y^2 + 2xy^3 dx = xy^2 + x^2y^3 + k(y).$$

Differentiating with respect to y, we get

$$\frac{\delta\phi}{\delta y} = 2xy + 3x^2y^2 + k'(y) = 2xy + 3x^2y^2 + 4y^3,$$

Thus $k'(y) = 4y^3$ and $k(y) = y^4 + c$. Putting $c = 0$, we find the potential

$$\phi(x, y) = xy^2 + x^2y^3 + y^4$$

(it is easily verified that we indeed have $\nabla\phi = \mathbf{F}$).

The work done by $\mathbf{F}$ on a particle moving on any path from $(1, -1)$ to $(2, 1)$ is

$$\phi(2,1)-\phi(1,-1) = 2\cdot 1^2+2^2\cdot 1^3+1^4-\left(1\cdot(-1)^2 + 1^2\cdot(-1)^3 + (-1)^4\right) = 7-1 = 6.$$