

Math 213: Calculus IV

June 14/01  
Quiz # 6, Solutions.



1 - Use Gauss' divergence theorem to evaluate

$$\int \int_{\Sigma} \mathbf{F} \cdot \mathbf{N} d\sigma$$

where  $\mathbf{F}(x, y, z) = \left(\frac{x^3}{3}, \frac{y^3}{3}, \frac{z^3}{3}\right)$  and  $\Sigma$  is the sphere  $\{(x, y, z) : x^2 + y^2 + z^2 = 4\}$ .

By Gauss' divergence theorem, we have

$$\int \int_{\Sigma} \left(\frac{x^3}{3}, \frac{y^3}{3}, \frac{z^3}{3}\right) \cdot \mathbf{N} d\sigma = \int \int \int_M (x^2 + y^2 + z^2) dx dy dz,$$

where  $M = \{(x, y, z) : x^2 + y^2 + z^2 \leq 4\}$  is the interior of  $\Sigma$ . With the spherical substitutions  $x = \rho \sin(\phi) \cos(\theta)$ ,  $y = \rho \sin(\phi) \sin(\theta)$ ,  $z = \rho \cos(\phi)$  and  $dx dy dz = \rho^2 \sin(\phi) d\rho d\phi d\theta$ , we get  $x^2 + y^2 + z^2 = \rho^2$  wence

$$\int \int \int_M (x^2 + y^2 + z^2) dx dy dz = \int_0^{2\pi} \int_0^{\pi} \int_0^2 \rho^2 \cdot \rho^2 \sin(\phi) d\rho d\phi d\theta = \frac{128\pi}{5}.$$

2 - Let  $\mathbf{F}(x, y, z) = (\sqrt{x^2 + 1}, \sqrt{y^2 + 1}, yz)$ . Evaluate  $\oint_C \mathbf{F} \cdot d\mathbf{C}$ , where  $C$  is the closed curve consisting of the line segments from  $(1, 0, 0)$  to  $(0, 1, 0)$ , from  $(0, 1, 0)$  to  $(0, 0, 1)$  and from  $(0, 0, 1)$  to  $(1, 0, 0)$  (oriented in that direction). (Hint:  $C$  is the boundary of the surface  $\Sigma$  defined by

$$\Sigma = \{(x, y, z) : x + y + z = 1, x \geq 0, y \geq 0, z \geq 0\}.$$

By Stoke's theorem, we have

$$\oint_C (\sqrt{x^2 + 1}, \sqrt{y^2 + 1}, yz) \cdot d\mathbf{C} = \int \int_{\Sigma} (\nabla \times (\sqrt{x^2 + 1}, \sqrt{y^2 + 1}, yz)) \cdot \mathbf{N} d\sigma.$$

The right side would take time to evaluate because it requires three separate line integrals, two of which involve intricate square root terms. This difficulty is avoided in the left side, because

$$\nabla \times (\sqrt{x^2 + 1}, \sqrt{y^2 + 1}, yz) = (z, 0, 0),$$

and this allows much simplification.

To complete the setup, we note that on  $\Sigma$ , we have  $\mathbf{N} = \frac{1}{\sqrt{3}}(1, 1, 1)$  and  $z = 1 - x - y = S(x, y)$ , with  $d\sigma = \sqrt{3} dy dx$ . The bounds for  $x$  are 0 and 1, and  $0 \leq y \leq 1 - x$  since  $z = 1 - x - y \geq 0$ . Therefore

$$\oint_C \mathbf{F} \cdot d\mathbf{C} = \int_0^1 \int_0^{1-x} (z, 0, 0) \cdot (1, 1, 1) dy dx = \int_0^1 \int_0^{1-x} (1 - x - y) dy dx = \frac{1}{6}.$$